

# Bose-Einstein condensate in non-homogeneous gravitational field

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## Abstract

*Ground state properties of trapped Bose condensate with repulsive interaction in non-homogeneous gravitational field are studied. Spatial structure of Bose condensate and its momentum distributions in 3-D anisotropic trap are considered by the solution of the modified non-linear Schrödinger equation. The results are compared with the corresponding properties of condensate in a harmonic trap without gravitational field.*

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Bose-Einstein condensation (BEC) is an outstanding example of a case where the properties of quantum matter can be studied in laboratory experiments. Since the theoretical discovery of BEC in the mid-1920s, most studies in this area were concentrated on the properties of homogeneous condensates with effectively infinite spatial extents. However, experimental success in achieving and observation of Bose condensation in trapped dilute atomic gases, which was done during the last five years, provided the possibility for study of the properties of an inhomogeneous confined condensate [1]-[3]. A trapped Bose gas exhibits features that are impossible to observe in a homogeneous medium. One of the remarkable features of these experiments is the observation of Bose condensation in coordinate as well as in momentum space. The localization in space of the condensed fraction of a low temperature Bose gas allows the detection of the condensate and to estimate the size of the BEC sample, to find its shape and momentum distribution of its atoms [4]-[5]. Moreover, the ability to manipulate such coherent matter provided additional stimuli for experimental and theoretical study of this phenomenon. The properties and behavior of a BEC

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in a trap depend on the presence of various factors. One of these factors is the interaction of the trapped atoms with a gravitational field. Gravity has a dominant influence on the evolution of atoms released from the trap. The dominance of gravity over the thermal effects has been successfully used in atom cloud gravimeters [6]-[7]. The use of BEC as a gravity and gravity gradient sensor opens wide perspectives for the creation of highly sensitive gravimeters and gravity gradiometers [8]. In this paper we present quantitative results on the influence of a non-homogeneous gravitational field on the behavior of a Bose condensate with repulsive interactions in a 3-D anisotropic trap. We determine the properties of a BEC in a gravitational field by studying its geometry, momentum distribution, and dynamical properties. We concentrate on the results obtained by the analysis of the solutions to the non-linear Schrödinger equation for trapped atoms in external gravitational field. Zero temperature solutions for the condensate are found using the Gross-Pitaevski (GP) equation [9]-[12]. Our objective is to provide a general framework for the analysis of the behavior of a confined Bose condensate which interacts with a gravitational field through the modified GP-equation and to investigate its properties.

A BEC created in a trap is under the influence of different forces, and the properties of the condensate depend on the combination of three potentials. A magnetic trap generates a time-averaged harmonic potential, while the second effective potential is formed from the binary collisions of the atoms. Finally, the condensate is under the influence of an external gravitational field which gives rise to a gravitational potential. The properties of the condensate at zero temperature are described by GP theory [13]-[15] which in its standard form includes two potentials. Our aim is to modify the GP equation in order to take into account gravitational interactions. The many body Hamiltonian describing  $N$  interacting bosons with atomic mass  $m$  is given by the equation

$$H = \int d\mathbf{x} \hat{\psi}^+(\mathbf{x}) \left[ -\frac{\hbar^2}{2m} \Delta + V_{ext}(\mathbf{x}) \right] \hat{\psi}(\mathbf{x}) + \frac{1}{2} \int d\mathbf{x}' d\mathbf{x} \hat{\psi}^+(\mathbf{x}') \hat{\psi}^+(\mathbf{x}) V(\mathbf{x}' - \mathbf{x}) \hat{\psi}(\mathbf{x}') \hat{\psi}(\mathbf{x}), \quad (1)$$

where  $\hat{\psi}(\mathbf{x})$  and  $\hat{\psi}^+(\mathbf{x})$  are the boson field operators which annihilate and create a particle at

the position  $\mathbf{x}$ , respectively, and  $V(\mathbf{x}' - \mathbf{x})$  is two body interatomic potential. Time evolution for the field operator  $\hat{\psi}(\mathbf{x}, t)$  is found from the Heisenberg equation

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{\psi}(\mathbf{x}, t) &= [\hat{\psi}(\mathbf{x}, t), H] \\ &= \left[ -\frac{\hbar^2}{2m} \Delta + V_{ext}(\mathbf{x}) \right. \\ &\quad \left. + \int d\mathbf{x}' \hat{\psi}^\dagger(\mathbf{x}', t) V(\mathbf{x}' - \mathbf{x}) \hat{\psi}(\mathbf{x}', t) \right] \hat{\psi}(\mathbf{x}, t). \end{aligned} \quad (2)$$

Only binary collisions at low energy are relevant for the dilute gas at low temperatures. These collisions are well described by s-wave scattering processes and allow us to replace  $V(\mathbf{x}' - \mathbf{x})$  with an effective interaction

$$V(\mathbf{x}' - \mathbf{x}) = V_o \delta(\mathbf{x}' - \mathbf{x}), \quad (3)$$

where the coupling constant  $V_o$  is related to s-wave scattering length  $a$  through

$$V_o = \frac{4\pi\hbar^2 a}{m}. \quad (4)$$

To overcome the problem of solving the full many-body Schrödinger equation for interacting systems we can use the mean field approach [16],[17]. The mean field approach is based on separation of the condensate fraction of the bosonic field operator. For a non-uniform and time-dependent configuration we have

$$\hat{\psi}(\mathbf{x}, t) = \phi(\mathbf{x}, t) + \hat{\psi}'(\mathbf{x}, t), \quad (5)$$

where  $\phi(\mathbf{x}, t)$  is a complex function defined as the expectation value for the field operator  $\phi(\mathbf{x}, t) = \langle \hat{\psi}(\mathbf{x}, t) \rangle$ . As a result, the equation for the condensate wave function  $\phi(\mathbf{x}, t)$  is found using (2) and (5), and is written in the form

$$i\hbar \frac{\partial}{\partial t} \phi(\mathbf{x}, t) = \left[ -\frac{\hbar^2}{2m} \Delta + V_{ext}(\mathbf{x}) + V_o |\phi(\mathbf{x}, t)|^2 \right] \phi(\mathbf{x}, t). \quad (6)$$

This non-linear Schrödinger type equation is called the Gross-Pitaevski (GP) equation. The validity of this equation is based on the conditions that the s-wave scattering length is smaller than the average distance between atoms, and that the number of atoms in the condensate

is large. To obtain the GP-equation for the ground state, one can write the condensate wave function as

$$\phi(\mathbf{x}, t) = \phi(\mathbf{x})e^{-i\mu t/\hbar}, \quad (7)$$

where the wave function  $\phi(\mathbf{x})$  is normalized to the total number of particles  $\int d\mathbf{x}|\phi(\mathbf{x})|^2 = N$ , and  $\mu$  is a chemical potential. Inserting (7) into (6), we find the equation for the wave function  $\phi(\mathbf{x})$

$$\left[ -\frac{\hbar^2}{2m}\Delta + V_{ext}(\mathbf{x}) + \frac{4\pi\hbar^2 a}{m}|\phi(\mathbf{x})|^2 \right] \phi(\mathbf{x}) = \mu\phi(\mathbf{x}). \quad (8)$$

The external potential for the confined atoms is the sum of the trap potential and a contribution due to the interaction with gravitational field:  $V_{ext} = V_{trap} + V_g$ . The magnetic trap generates a harmonic confining potential of the form

$$V_{trap}(\mathbf{x}) = \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2), \quad (9)$$

where  $\omega_i$  are the projections of the angular trap frequencies on the coordinate axis, and vector  $\mathbf{x} = (x, y, z)$  defines the displacement from the trap center  $\mathbf{x}_o = (x_o, y_o, z_o)$ . The gravitational interaction is described by  $V_g(\mathbf{X}) = m\Phi(\mathbf{X})$ , where the potential for the given mass distribution is found as  $\Phi(\mathbf{X}) = -G \sum_i m_i / |\mathbf{X}_i - \mathbf{X}|$ . The potential of a gravitational field is an unobservable value. The intrinsic characteristics that uniquely characterize the gravitational field in Newtonian gravity are the first and the second spatial derivatives of the potential. They form the vector of gravitational acceleration  $g_i = -\partial\Phi/\partial x_i$ , and the gravity gradient (or Eötvös) tensor

$$\Gamma_{jk} = \frac{\partial^2 \Phi}{\partial x^j \partial x^k}. \quad (10)$$

Each component of this tensor defines the acceleration difference along the direction  $j$  per unit separation along the direction  $k$ . The gravity gradient tensor is symmetric, and its trace is related to the local mass density by Poisson's equation. These conditions define the number of its independent components. In the local coordinate basis of the trap the gravitational potential can be written with the help of the vector gravitational acceleration and the gravity gradient tensor as

$$\Phi(\mathbf{x}_o, \mathbf{x}) = \Phi(\mathbf{x}_o) - g_j(\mathbf{x}_o)x^j + \frac{1}{2}\Gamma_{jk}(\mathbf{x}_o)x^j x^k. \quad (11)$$

We will locate the coordinate system of the trap in such a way as to obtain the components of the vector of gravitational acceleration as

$$g_x = 0, \quad g_y = 0, \quad g_z = -|\mathbf{g}|, \quad (12)$$

In this approximation, the leading components of the gravity gradient tensor are  $\{\Gamma_{xx}, \Gamma_{yy}, \Gamma_{zz}\}$ . The components of the gravity gradient are not independent. They are connected with each other by the Laplace equation which leads to  $\Gamma_{xx} + \Gamma_{yy} = -\Gamma_{zz}$ . In the given coordinate system, the equation for the external potential is

$$V_{ext}(\mathbf{x}) = \frac{m}{2} \left[ (\omega_x^2 + \Gamma_{xx}) x^2 + (\omega_y^2 + \Gamma_{yy}) y^2 + (\omega_z^2 + \Gamma_{zz}) z^2 \right] + mgz + m\Phi_o. \quad (13)$$

Consider an axially symmetric confined potential with angular frequencies  $\omega_x = \omega_y = \omega_\perp$  and  $\omega_z = \lambda\omega_\perp$ , where  $\lambda$  is an asymmetry parameter. Then, the symmetry of the trap allows us to write down GP equation as

$$\left[ -\frac{\hbar^2}{2m} \Delta + \frac{m\omega_\perp^2}{2} (\gamma_x^2(\Gamma)x^2 + \gamma_y^2(\Gamma)y^2 + \gamma_z^2(\Gamma)z^2) + mgz + \frac{4\pi\hbar^2 a}{m} |\phi(\mathbf{x})|^2 \right] \phi(\mathbf{x}) = (\mu_g - m\Phi_o) \phi(\mathbf{x}), \quad (14)$$

where

$$\begin{aligned} \gamma_x^2(\Gamma) &= \left( 1 + \frac{\Gamma_{xx}}{\omega_\perp^2} \right), \\ \gamma_y^2(\Gamma) &= \left( 1 + \frac{\Gamma_{yy}}{\omega_\perp^2} \right), \\ \gamma_z^2(\Gamma) &= \lambda^2 \left( 1 + \frac{\Gamma_{zz}}{\omega_z^2} \right). \end{aligned} \quad (15)$$

In the equation (15), the factors  $\{\gamma_x(\Gamma), \gamma_y(\Gamma), \gamma_z(\Gamma)\}$ , are functions of the diagonal components of gravity gradient tensor. In order to simplify the analysis of the equation (14), we will scale the coordinates using a spatial scale parameter  $\zeta = (\hbar/m\omega_\perp)^{1/2}$  :

$$\mathbf{x}_1 = \zeta^{-1} \mathbf{x}. \quad (16)$$

We also introduce the scaled wave function

$$\phi(\mathbf{x}) = \left(N\zeta^{-3}\right)^{\frac{1}{2}} \phi_1(\mathbf{x}_1), \quad (17)$$

which satisfies the normalization condition

$$\int d^3\mathbf{x}_1 |\phi_1(\mathbf{x}_1)|^2 = 1. \quad (18)$$

In the new variables, the dimensionless GP-equation (14) has the form

$$\begin{aligned} & \left[ -\Delta_1 + \gamma_x^2(\Gamma)x_1^2 + \gamma_y^2(\Gamma)y_1^2 + \gamma_z^2(\Gamma)z_1^2 + 2D(g)z_1 \right. \\ & \left. + u_1 |\phi_1(x_1, y_1, z_1)|^2 \right] \phi_1(x_1, y_1, z_1) = 2\mu_1 \phi_1(x_1, y_1, z_1), \end{aligned} \quad (19)$$

where

$$u_1 = \frac{8\pi a N}{\zeta} \quad (20)$$

is the dimensionless interaction strength,

$$\mu_1 = \frac{1}{\hbar\omega_\perp} (\mu_g - m\Phi_o) \quad (21)$$

is a dimensionless eigenvalue of the equation, and

$$D(g) = \frac{mg\zeta}{\hbar\omega_\perp} = g \left( \frac{m}{\hbar\omega_\perp^3} \right)^{\frac{1}{2}} \quad (22)$$

is a linear function of gravity. The dimensionless form of GP equation (19) is suitable for integration by means of appropriate methods. The non-linear term in this equation does not allow us to obtain analytical solutions, and numerical computer simulation has to be used instead. However, it is possible to find analytical solutions in two limiting cases: non-interacting particles and strongly repulsive interactions.

(a) *Condensate without interaction*

If we neglect two-body collisions, solving the GP equation can be reduced to finding the solution of (19) with  $u_1 = 0$

$$\begin{aligned} & \left[ -\Delta_1 + \gamma_x^2(\Gamma)x_1^2 + \gamma_y^2(\Gamma)y_1^2 + \gamma_z^2(\Gamma) \left( z_1 + \frac{D(g)}{\gamma_z^2(\Gamma)} \right)^2 \right] \phi_1(x_1, y_1, z_1) \\ & = 2\tilde{\mu}_1 \phi_1(x_1, y_1, z_1), \end{aligned} \quad (23)$$

where

$$\tilde{\mu}_1(g, \Gamma) = \mu_1 + \frac{1}{2} \left( \frac{D(g)}{\gamma_z(\Gamma)} \right)^2. \quad (24)$$

The dimensionless ground state solution is

$$\begin{aligned} \phi_1(x_1, y_1, z_1) = A \exp & \left[ -\frac{\gamma_x(\Gamma)x_1^2}{2} - \frac{\gamma_y(\Gamma)y_1^2}{2} \right] \times \\ & \exp \left[ -\frac{\gamma_z(\Gamma)}{2} \left( z_1 + \frac{D(g)}{\gamma_z^2(\Gamma)} \right)^2 \right], \end{aligned} \quad (25)$$

where the normalization constant is found from condition (18) as  $A = (\gamma_x \gamma_y \gamma_z / \pi^3)^{1/4}$ , and

$$\tilde{\mu}_1(g, \Gamma) = (\gamma_x(\Gamma) + \gamma_y(\Gamma) + \gamma_z(\Gamma)) / 2. \quad (26)$$

The ratio of the mean values of momenta can be found using the wave function

$$\phi_1(p_x, p_y, p_z) = \exp \left[ -\frac{\zeta^2}{2\hbar^2} \left( \gamma_x^{-1}(\Gamma)p_x^2 + \gamma_y^{-1}(\Gamma)p_y^2 + \gamma_z^{-1}(\Gamma)p_z^2 \right) \right]. \quad (27)$$

From this it follows that the ratios of the mean momenta are

$$\begin{aligned} \sqrt{\frac{\langle p_z^2 \rangle}{\langle p_x^2 \rangle}} &= \sqrt{\frac{\gamma_z(\Gamma)}{\gamma_x(\Gamma)}}, & \sqrt{\frac{\langle p_z^2 \rangle}{\langle p_y^2 \rangle}} &= \sqrt{\frac{\gamma_z(\Gamma)}{\gamma_y(\Gamma)}}, \\ & & \sqrt{\frac{\langle p_x^2 \rangle}{\langle p_y^2 \rangle}} &= \sqrt{\frac{\gamma_x(\Gamma)}{\gamma_y(\Gamma)}}. \end{aligned} \quad (28)$$

As consequence, the anisotropy of the velocity distribution depends on the gravity gradient. As the Bose condensate is characterized by the absence of thermal excitations, equations (28) reflect the result of atomic motion in the trapping potential only. The transverse and vertical widths of the Gaussian distribution (25) are

$$\langle x_1^2 \rangle = \frac{1}{2\gamma_x(\Gamma)}, \quad \langle y_1^2 \rangle = \frac{1}{2\gamma_y(\Gamma)}, \quad \langle z_1^2 \rangle = \frac{1}{2\gamma_z(\Gamma)}, \quad (29)$$

and depend on the gravity gradient. The average  $\langle z_1 \rangle = -D(g)/\gamma_z^2(\Gamma)$  defines the shift of the condensate cloud along the  $z$ -axis.

*(b) Condensate in a repulsive limit*

The parameter  $u_1$  of equation (19) is proportional to the total number of atoms in the condensate.

In the limit of large  $N$  one can neglect the kinetic energy term (the so-called Thomas-Fermi approximation) and rewrite this equation as

$$\begin{aligned} & \left[ \gamma_x^2(\Gamma)x_1^2 + \gamma_y^2(\Gamma)y_1^2 + \gamma_z^2(\Gamma)z_1^2 + 2D(g)z_1 \right. \\ & \left. + u_1|\phi_1(x_1, y, z_1)|^2 \right] \phi_1(x_1, y, z_1) = 2\mu_1\phi_1(x_1, y_1, z_1). \end{aligned} \quad (30)$$

The ground state solution of equation (30) is

$$|\phi_1(x_1, y_1, z_1)|^2 = \frac{1}{u_1} \left[ 2\tilde{\mu}_1 - \gamma_x^2(\Gamma)x_1^2 - \gamma_y^2(\Gamma)y_1^2 - \gamma_z^2(\Gamma) \left( z_1 + \frac{D(g)}{\gamma_z^2(\Gamma)} \right)^2 \right], \quad (31)$$

where  $\tilde{\mu}_1$  is found from the normalization condition (18) as

$$\begin{aligned} 2\tilde{\mu}_1(g, \Gamma) &= \left[ \frac{15}{8\pi} (\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma)) u_1 \right]^{2/5} \\ &= \left[ \frac{15aN}{\zeta} (\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma)) \right]^{2/5}. \end{aligned} \quad (32)$$

The chemical potential depends on the trapping frequencies entering the potential  $V_{trap}$ , the number of particles  $N$ , and the components of the gravity gradient. The right hand side of equation (31) defines the area of the condensate location. The boundary of the condensate cloud has the form of an ellipsoid with semiaxes

$$R_x(g, \Gamma) = \frac{R}{\gamma_x(\Gamma)}, \quad R_y(g, \Gamma) = \frac{R}{\gamma_y(\Gamma)}, \quad R_z(g, \Gamma) = \frac{R}{\gamma_z(\Gamma)}, \quad (33)$$

where  $R = \sqrt{2\tilde{\mu}_1(g, \Gamma)}$ . As follows from equation (31), the center of the ellipsoid is shifted along the  $z$ -axis by  $-D(g)/\gamma_z^2(\Gamma)$ .

In the conclusion we will review the obtained results. The expression for wave function (25) in the the approximation of weak interaction between atoms allows us to find statistical characteristics of the density distribution (such as first and second cumulants, moments, mean and variance along each axis). As follows from (25) there is a normal distribution along the  $z$ -axis as well as along the  $x$  and  $y$ -axis. The distribution along the  $z$ -axis is characterized by the moment  $m_1 = -D(g)/\gamma_z^2(\Gamma)$ , and dispersion  $\sigma_z^2 = (2\gamma_z(\Gamma))^{-1}$ . Therefore, the dispersion  $\sigma_z^2$  varies with respect to the  $\Gamma_{zz}$  component of the gravity gradient tensor, and the density distribution



is broadened in the  $z$  direction due to tidal forces. The condensate is shifted in proportion to the gravitational acceleration by  $\langle z_1 \rangle = m_1 = -D(g)/\gamma_z^2(\Gamma)$ . Similarly, the dispersions of the distributions in  $x$  and  $y$  are given by  $\sigma_x^2 = (2\gamma_x(\Gamma))^{-1}$  and  $\sigma_y^2 = (2\gamma_y(\Gamma))^{-1}$ . This shows that the density distribution is squeezed in the transverse direction due to the gravitational interaction of the atoms with the gravitational field. The central density of this distribution is  $n(0) = |\phi(0)|^2 = (N/\zeta^3) [(\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma))/\pi^3]^{1/2}$  and depends on the number of particles in the condensate  $N$  and the components of the gravity gradient through the  $\gamma$ -factors. The anisotropy in the velocity distributions  $\sqrt{\langle v_z^2 \rangle / \langle v_x^2 \rangle} = \sqrt{\gamma_z(\Gamma)/\gamma_x(\Gamma)}$  and  $\sqrt{\langle v_z^2 \rangle / \langle v_y^2 \rangle} = \sqrt{\gamma_z(\Gamma)/\gamma_y(\Gamma)}$  for non-interacting condensates is given by the ratio of  $\gamma$ -factors (Eq. 28), and therefore is connected with the variation of the diagonal components of the gravity gradient. In spite of the axial symmetry of the trap potential, there is an asymmetry in the ratio  $\sqrt{\langle v_y^2 \rangle / \langle v_x^2 \rangle}$ , which is given by the ratio of  $\gamma$ -factors  $\sqrt{\gamma_y(\Gamma)/\gamma_x(\Gamma)}$ . As follows from equation (31), the shape of the condensate in the strongly repulsive limit also depends on the diagonal components of the gravity gradient tensor. In the strongly repulsive limit, the 3-D shape of the condensate is an ellipsoid with semiaxes given by equation (33). The ellipsoid is squeezed in the transverse ( $x - y$ ) plane, and extended in the axial ( $z$ ) direction due to the contributions of the gravity gradient. The ellipsoid is also shifted in the  $z$  direction due to gravity. The energy per particle can be found from the equation  $\tilde{\mu} = \partial E / \partial N$  and is given by  $E/N = 2/7\tilde{\mu}$ , where the dimensional chemical potential is  $\tilde{\mu} = (\hbar\omega_\perp/2) [15aN(\gamma_x(\Gamma)\gamma_y(\Gamma)\gamma_z(\Gamma))/\zeta]^2/5$ . The contributions to the energy  $E$  in this approximation are related to the interaction between particles, the gravitational interaction and oscillator energies. The density of particles  $n(x, y, z) = |\phi(x, y, z)|^2$  is found directly from equation (31). As follows from equation (31), the density distribution vanishes in classical turning points on the surface of ellipsoid. The central density is  $n(0) = 2\tilde{\mu}_1 N / (\zeta^3 u_1)$ , and depends on the number of particles and the gravity gradient. The size of the ellipsoid depends on the number of particles  $N$  (the semiaxis scale as  $N^{1/5}$ ) and the components of the gravity gradient. The components of the gravity vector and the gravity gradient tensor are defined by the source of the gravitational field. For example, for Earth gravity ( $g = 9.87 \text{ ms}^{-2}$  and  $\Gamma_{zz} = 3 \times 10^{-6} \text{ s}^{-2}$ ),

the correction for  $\gamma_z(\Gamma)$  depends on the frequency  $\omega_z$  (Eq.15) and can be numerically estimated as  $\Gamma_{zz}/\omega_z^2 \sim 10^{-8}$  for  $\omega_z \sim 20Hz$ .

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